Fundamental limits on isoplanatic correction with multiconjugate adaptive optics

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Received December 2, 2002; revised manuscript received April 22, 2003; accepted May 28, 2003

We investigate the performance of a general multiconjugate adaptive optics (MCAO) system in which signals from multiple reference beacons are used to drive several deformable mirrors in the optical beam train. Taking an analytic approach that yields a detailed view of the effects of low-order aberration modes defined over the metapupil, we show that in the geometrical optics approximation, N deformable mirrors conjugated to different ranges can be driven to correct these modes through order N with unlimited isoplanatic angle, regardless of the distribution of turbulence along the line of sight. We find, however, that the optimal deformable mirror shapes are functions of target range, so the best compensation for starlight is in general not the correction that minimizes the wave-front aberration in a laser guide beacon. This introduces focal anisoplanatism in the wave-front measurements that can be overcome only through the use of beacons at several ranges. We derive expressions for the number of beacons required to sense the aberration to arbitrary order and establish necessary and sufficient conditions on their geometry for both natural and laser guide stars. Finally, we derive an expression for the residual uncompensated error by mode as a function of field angle, target range, and MCAO system geometry.

OCIS codes: 010.1080, 010.1330, 010.7350, 110.6960.

1. INTRODUCTION

All current telescopes in the 6- to 10-m class now include or will include adaptive optics (AO), thus allowing huge improvements in sensitivity and resolution. The potential gains will be even higher for the next generation of telescopes 20 m or more in diameter.[1–3] But just as our telescopes take the next big step forward, so too must the AO to realize these gains. Single sodium beacons, for example, are not adequate to correct even a narrow field of view at such large aperture sizes because of focus anisoplanatism. The goal will be to provide imaging at the diffraction limit of resolution, now sharper than ever, over a wider field and at shorter wavelengths than our current tools can manage. This must all be done while maintaining a point-spread function that is as stable as possible in both field of view and time.

To achieve this goal, multiconjugate adaptive optics (MCAO), originally proposed by Beckers,[4] will be required, and indeed the first MCAO systems for deployment at astronomical telescopes are now being designed.[5–7] MCAO broadens the corrected field of view by reproducing the three-dimensional structure of the atmospheric phase aberrations in a series of deformable mirrors (DMs) in the telescope optics. Path-length errors in rays coming from a range of angles are thereby compensated, with the angular size of this isoplanatic field limited by the fidelity of the reproduction in the direction of the optical axis and hence the number of DMs.

Previous work has explored the characteristics of MCAO correction from knowledge of the statistical properties of the atmospheric aberration.[8–11] In this paper we develop an analytic approach, with the analysis supported and extended by numerical computation. The detailed view of the effect of phase aberration and compensation that this affords contrasts with the necessarily averaged view offered by statistical approaches and leads to new insights into the abilities and requirements of an MCAO system.

We find that aberration over the metapupil to any given order can be compensated with no isoplanatic error for targets at a common distance by using a number of DMs equal to the order number and for all targets in the volume with the addition of one more DM. Significantly, we find that in a high-order MCAO system, the DM shapes that minimize wave-front aberration in starlight are not those that would drive laser guide star (LGS) wave-front sensor (WFS) signals to null. This difference represents a very significant error term in any future scheme to determine absolute tilt from LGS measurements. We also establish the requirements on the number and geometry of reference beacons to sense aberration to arbitrary order, for cases relying on both natural guide stars (NGSs) and laser beacons (LGSs). The same work leads to an expression for the elements of the modal influence function matrix, whose inverse is the tomographic reconstructor relating the WFS signals to the corrections sent to the DMs. We leave aside for now the question of the selection of DM conjugate heights, although, in principle, this can be addressed by examining the WFS signals from multiple guide stars with an algorithm similar to that used in scintillation detection and ranging (SCIDAR) measurements.

2. WAVE-FRONT CORRECTION

In general, when wave-front phase errors in light traversing a turbulent medium from a point source are corrected in the pupil plane of an optical system, phase errors in the

1084-7529/2003/101949-09$15.00

light from any point nearby in the field are not perfectly corrected, because the light has followed a different path. If, instead, the phase delay at each point in the atmosphere within some field of view could be reproduced with opposite sign at a conjugate point in a three-dimensional correcting optic, perfect imaging would be restored across the field. It is not clear, though, that this distribution of phase compensation in the volume is the only solution that restores perfect imaging. Here we investigate the requirements that must be satisfied by the correcting optics to obtain a solution and, in particular, the degree of compensation that can be expected when the ideal three-dimensional corrector is approximated by a small number of DMs conjugated to distinct planes in the atmosphere.

In the following analysis we assume the geometrical-optics approximation in which phase errors are accumulated along a ray path without regard to refraction. This assumption is reasonable for the weak turbulence typically encountered in near-vertical observations at a good astronomical site.[12]

A. Choice of Basis Set

We assume that the phase is a well-behaved function, i.e., continuous and single valued. This restricts us to the domain of weak turbulence: In strong turbulence it is quite possible for phase wraps to occur.[13]

Most work on the analysis of atmospheric phase perturbations relies on the Zernike basis set to describe the aberrated phase front $\phi(\rho, \theta)$. A complete description is given by coefficients ζ_{nm} of the Zernike polynomials Z_{nm}, where the radial and azimuthal orders of the polynomial are, respectively, n and $n - 2m$:

$$\phi(\rho, \theta) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \zeta_{nm} Z_{nm}(\rho, \theta). \tag{1}$$

Zernike's functions have the very convenient property of orthonormality over the unit circle, and the low-order functions are easily related to classical aberrations that opticians learn to recognize. They are less useful in the present context, however, in which we must consider aberrations over a smaller window at an arbitrary location within the unit circle. This geometry arises from the footprint of beacon light approaching the telescope on a metapupil whose diameter is determined at a given height by the diameter of the telescope and the field of view.

Instead, we follow Ragazzoni *et al.*[14] and choose a simple polynomial representation $\phi(x, y)$ in rectangular coordinates as the most convenient for the present analysis. Recognizing that Z_{nm} can be written as a polynomial of degree n in x and y, we can approximate any function ϕ through order N by

$$\phi \approx \sum_{n=0}^{N} \sum_{m=0}^{n} \zeta_{nm} Z_{nm} \equiv \sum_{i=0}^{N} \sum_{j=0}^{N-i} \alpha_{ij} x^i y^j, \tag{2}$$

where the coordinates x and y are expressed in units of the telescope radius R. The matrix relating ζ_{nm} and α_{ij} is square, nonsingular, and straightforward to compute.[9,14]

B. Compensation of Stellar Wave Fronts

Unlike a classical single-conjugate AO system, we are concerned with phase aberrations inside a truncated cone that defines the field of view rather than a cylinder defined by light from a single star. We start then by decomposing the phase delay introduced per unit height across the metapupil at height h into modes $x^i y^j$ with coefficients $\alpha_{ij}(h)$.

The atmospheric phase in the telescope pupil accumulated by light from a star at angular position (ξ, v) in the field may be written as a polynomial in four variables:

$$\Phi_*(x, y; \xi, v) = \int_0^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{ij}(h) X^i Y^j \mathrm{d}h. \tag{3}$$

It will be convenient to express quantities of height normalized by R, in which case we shall write, for instance, $\hat{h} = h/R$. We set $X = x + \xi\hat{h}$ and $Y = y + v\hat{h}$ as the coordinates at height h of a ray coming from (ξ, v) and intersecting the pupil at point (x, y). Note that Eq. (3) contains a piston term $i = j = 0$, which for notational convenience we leave in for the time being.

In general, our MCAO system will have only a finite number of degrees of freedom, chosen to balance fitting error against other sources of residual aberration. We then write $\Phi_*{}^N$ as the component of the phase aberration containing modes from orders 1 through N that we can hope to correct:

$$\Phi_*{}^N(x, y; \xi, v) = \int_0^{\infty} \sum_{i=0}^{N} \sum_{j=0}^{N-i} \alpha_{ij}(h) X^i Y^j \mathrm{d}h. \tag{4}$$

Expanding the X^i and Y^j, we find

$$\Phi_*{}^N(x, y; \xi, v) = \sum_{i=0}^{N} \sum_{k=0}^{i} \binom{i}{k} x^{i-k} \xi^k \sum_{j=0}^{N-i} \sum_{l=0}^{j} \binom{j}{l} y^{j-l} v^l \int_0^{\infty} \alpha_{ij} \hat{h}^{k+l} \mathrm{d}h, \tag{5}$$

$$= \sum_{k=0}^{N} \sum_{l=0}^{N-k} x^k y^l \sum_{i=k}^{N-l} \binom{i}{k} \xi^{i-k} \sum_{j=l}^{N-i} \binom{j}{l} v^{j-l} \times \int_0^{\infty} \alpha_{ij} \hat{h}^{i+j-k-l} \mathrm{d}h, \tag{6}$$

where $\binom{i}{k}$ is the binomial coefficient $i!/k!(i-k)!$.

Equation (6) quantifies the effect that an atmospheric mode of given degree has on starlight. We note in particular that each mode (i, j) arising at $\hat{h} > 0$ gives rise to aberrations (k, l) of equal and lower degrees in the wave front of an off-axis star. This will be of importance when we examine the requirements on beacon geometry for wave-front sensing. A well-known example is the breathing mode, or radial motion of off-axis stellar images in compensated imaging from single-conjugate AO systems, caused by uncorrected field-dependent tilt terms arising from high-altitude aberrations of second order and above.[11]

The phase applied by a given DM in the beam train is a function of two variables only, namely, x and y. If we apply a general phase function of order N_m to mirror m, we can write the function as

$$\psi_m(x, y) = \sum_{i=0}^{N_m} \sum_{j=0}^{N_m-i} \beta_{ij,m} x^i y^j. \quad (7)$$

Let a set of mirrors $m = 1...M$ be conjugated to heights H_m in the optical beam train. The correction phase Ψ applied by these mirrors in direction (ξ, v) is

$$\Psi(x, y; \xi, v) = \sum_{m=1}^{M} \psi_m(x + \xi \hat{H}_m, y + v \hat{H}_m). \quad (8)$$

If for simplicity we set $N_m = N$ for all m, that is, all the mirrors correct to the same degree, then

$$\Psi(x, y; \xi, v) = \sum_{k=0}^{N} \sum_{l=0}^{N-k} x^k y^l \sum_{i=k}^{N-l} \binom{i}{k} \xi^{i-k} \sum_{j=l}^{N-i} \binom{j}{l} \times v^{j-l} \sum_{m=1}^{M} \beta_{ij,m} \hat{H}_m{}^{i+j-k-l}. \quad (9)$$

The coefficients $\beta_{ij,m}$ are to be chosen as far as possible to cancel the atmospheric aberration for stars everywhere in the field of view simultaneously. Ideally, we wish to minimize the quantity $\langle(\Phi_* - \Psi)^2\rangle$, where $\langle\ldots\rangle$ indicates averaging over the aperture, the field of view, and time. Computing this minimum has been the subject of much work in statistical analysis and Monte Carlo simulation (see, for example, Refs. 8–11 and 15–17). Analytically, the first two averages are straightforward to compute for reasonably shaped apertures and a circular field. The third, on the other hand, involves the covariances of the height moments of the closed-loop aberration coefficients. Computing these appears feasible but remains a work in progress.

We assume for now a more modest goal: We seek conditions under which $\beta_{ij,m}$ can be found such that $\Psi = \Phi_*{}^N$; that is, the atmospheric phase error to Nth order, defined over the metapupil, is fully corrected at all field angles. Equating coefficients of like terms for both spatial and angular coordinates in Eqs. (6) and (9), we can say

$$\sum_{m=1}^{M} \beta_{ij,m} \hat{H}_m{}^n = \int_0^\infty \alpha_{ij} \hat{h}^n \mathrm{d}h,$$

where

$$0 \leq n \leq i + j. \quad (10)$$

Each of these simultaneous equations constrains the corresponding nth-degree aberration introduced in off-axis starlight by mode $x^i y^j$. Furthermore, for the highest-order atmospheric aberrations we are attempting to correct, where $i + j = N$, Eq. (10) represents a set of $N + 1$ simultaneous equations. A complete solution then is possible, provided that there are exactly $N + 1$ variables $\beta_{ij,m}$, or $M = N + 1$.

However, the constraint imposed by $n = N$, implying k, $l = 0$ in Eqs. (6) and (9), merely represents piston and is of no account in terms of the imaging system's performance. Ignoring the corresponding equation also in Eq. (10) reduces the number of independent atmospheric integrals, and therefore the requirement on M, from $N + 1$ to N.

We have made no assumptions in this analysis about the vertical distribution of turbulence or the shape of the aperture. We conclude therefore that a MCAO system with N mirrors can compensate starlight for atmospheric aberration across the metapupil to Nth order with no isoplanatic error under any $C_n{}^2$ profile. An inductive argument and the principle of linear superposition then show that this arrangement is also sufficient to correct all lower orders simultaneously. Strictly, this is true for any set of DM conjugates H_m, provided only that they are all distinct; that is, no two DMs are conjugated to the same height.

Practical MCAO systems will contain no more than a few DMs. We therefore cannot expect to control more than the lowest orders in this way. In fact, as the order number increases, modes rapidly become poorly constrained: The DMs implement N constraints on each mode; yet for full correction, a mode of order $n > N$ requires n constraints and therefore will inevitably introduce anisoplanatism. We investigate this further in Section 4.

Since, by hypothesis, we can apply all modes to each DM, we, in fact, have an excess of degrees of freedom for controlling modes of order $< N$. The tomographic reconstructor matrix, which maps beacon measurements onto the DMs, must reject these excess modes to maintain a stable system. How many are there? Exactly n DMs are needed for fully isoplanatic control of modes of order n, of which there are $n + 1$ per DM. If we have N DMs, then we must reject $N(n + 1) - n(n + 1) = (N - n) \times (n + 1)$ modes. The total number to be removed, including piston modes, is then

$$\sum_{n=0}^{N} (N - n)(n + 1) = \frac{1}{6} N(N + 1)(N + 2). \quad (11)$$

C. Extension to Objects at Finite Range

In Subsection 2.B we assumed that the wave front originated from a source at infinity, an assumption that we now relax. The following analysis is cast in terms of a constellation of LGSs, since this represents the typical situation expected of an astronomical MCAO system. The results, however, apply generally to any object at finite range.

Let there be a constellation of laser beacons, indexed by b, launched from behind the telescope's secondary mirror. Each beacon propagates to height h_b along direction (ξ_b, v_b) with respect to the telescope's optical axis. The phase error Φ_b accumulated by a ray returning to point (x, y) in the pupil is

$$\Phi_b(x, y; \xi_b, v_b) = \int_0^{h_b} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha_{ij}(h) X_b{}^i Y_b{}^j \mathrm{d}h, \quad (12)$$

where $X_b = x(1 - h/h_b) + \xi_b \hat{h}$ and $Y_b = y(1 - h/h_b) + v_b \hat{h}$. For a natural star, $h_b \to \infty$, and Eq. (12) becomes identical to Eq. (3).

The correction phase seen by light from the beacon is

$$\Psi_b(x, y; \xi_b, v_b) = \sum_{m=1}^{M} \psi_m[x(1 - H_m/h_b) + \xi_b \hat{H}_m, \; y(1 - H_m/h_b) + v_b \hat{H}_m]. \quad (13)$$

Expanding Eqs. (12) and (13) and equating coefficients as before, we find

$$\int_0^{h_b} \alpha_{ij}(1 - h/h_b)^{i+j-n} \hat{h}^n \mathrm{d}h = \sum_{m=1}^{M} \beta_{ij,m}(1 - H_m/h_b)^{i+j-n} \hat{H}_m{}^n,$$

where

$$0 \leqslant n \leqslant i + j. \quad (14)$$

In the previous case, Eq. (10), the highest moment could be discarded because it constrained only piston. A similar piston constraint applies here for finite range but is now represented by a linear combination of the analogous Eq. (14). It remains possible therefore to remove the piston constraint and to correct metapupil aberrations in beacon light to Nth order with no isoplanatic error by use of N DMs. The solution, however, is different from that for starlight.

The difference is important because it means that the optimal correction for imaging astronomical objects at infinity is not the solution that drives WFS signals from a LGS to null. A simple one-dimensional example serves as an illustration. Referring to Fig. 1, we construct a geometry with two DMs compensating for quadratic aberration $\Phi = \alpha_2 x^2$ arising at some arbitrary height h_A. The aberration and compensation are sampled by light from a beacon at h_b and angle ξ_b.

The phase error accumulated along a ray between the beacon and a point x_0 in the pupil is $\phi_{\mathrm{err}} = \alpha_2 x_A{}^2 - \beta_{2,1} x_1{}^2 - \beta_{2,2} x_2{}^2$, where $x_A = \xi_b h_A + x_0(1 - h_A/h_b)$ and $x_n = \xi_b H_n + x_0(1 - H_n/h_b)$. Setting $H_1 = 0$ for convenience and requiring ϕ_{err} to be independent of x_0 (thus allowing only piston terms to remain) lead to the following values for the compensating mirror coefficients:

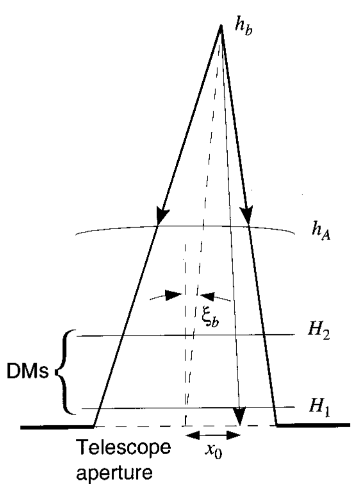

Fig. 1. Geometry of an example in which two DMs are able to correct quadratic aberration in wave fronts of objects at a particular range and anywhere in the field of view.

$$\beta_{2,1} = \alpha_2(1 - h_A/H_2)(1 - h_A/h_b), \qquad \beta_{2,2} = \alpha_2 h_A(h_b - h_A)/H_2(h_b - H_2). \quad (15)$$

This solution requires knowledge of the strength and height of the aberration, to be provided by the tomographic WFS, but is independent of x_0 and ξ_b. It therefore correctly compensates all rays intersecting the pupil from objects at all field angles. However, the solution is not independent of h_b, so the optimal mirror coefficients for correcting starlight, where $h_b \to \infty$, are different from those needed to correct the light from a LGS at finite range.

The aberration can in fact be corrected for all ranges through the use of one more DM, that is, allowing correction of Nth-order modes with $N + 1$ mirrors. In that case, Eqs. (14) are all solvable simultaneously, and corrections can be found that remove all Nth-order wave-front error from all objects regardless of range or position in the field.

We explore the practical effects of this difference in Section 4, but we note here that for the case of a single conjugate AO system using a single LGS, the effect contributes to focal anisoplanatism in a way that has nothing to do with regions of unsensed aberration outside the cone of returning LGS light. It is intrinsic to the use of a beacon at a range different from the target object. As such, it cannot be overcome by relying on multiple LGSs to fill the volume of atmosphere sampled by natural starlight.[18]

The range dependence illustrated above with a second-order aberration applies generally to all orders. Indeed, a similar analysis demonstrates that the global image motion of a scene, arising from the tilt modes over the metapupil, is range dependent. Although this is unimportant for current astronomical MCAO system designs using LGSs, because the lasers are not used to sense global tilt, it will be a concern in any system designed to recover tilt from laser signals in the future[19–22] and for any nonastronomical system viewing targets at finite distance. If image motion information is sensed from beacons at two distinct ranges, then tip–tilt driving signals can be computed for two DMs that would correct the motion independently of range. On the other hand, if a number of tilt-sensitive sodium LGSs were used to correct the motion of starlight in a 10-m telescope with a single DM conjugated to the ground, then under average turbulent conditions one calculates the range difference to contribute a residual wave-front error of ~200-nm root-mean-square (RMS). This would likely be the largest single error term in the compensated wave front.

3. SENSING THE WAVE FRONT

A pattern of reference beacons for wave-front sensing must be chosen that encodes the tomographic information needed to drive the DMs. We examine here the geometrical constraints that must be satisfied by the beacons to allow complete sensing of aberration to arbitrary order, treating both NGSs and LGSs. We assume that the individual wave fronts from each beacon have already been reconstructed in the conventional way, without regard to the finite range in the case of LGSs, and that we must now reconstruct the tomography.

A. Influence Function Matrix Elements

Ragazzoni *et al.*[14] first pointed out the possibility of building a tomographic reconstructor matrix that could compute directly the modal commands to the DMs on the basis of modal vectors from the WFSs. With the results from Section 2, we can write the elements of the influence matrix from which such a reconstructor would be derived. The correction phase Ψ_b, Eq. (13), seen by each beacon can also be written in terms of coefficients $\gamma_{kl,b}$ measured by the associated WFS:

$$\Psi_b(x, y; \xi_b, v_b) = \sum_{k=0}^{N} \sum_{l=0}^{N-k} \gamma_{kl,b} x^k y^l. \tag{16}$$

One then finds

$$\gamma_{kl,b} = \sum_{i=k}^{N-l} \binom{i}{k} \xi_b{}^{i-k} \sum_{j=l}^{N-i} \binom{j}{l} v_b{}^{j-l} \times \sum_{m=1}^{M} \beta_{ij,m} (1 - H_m/h_b)^{k+l} \hat{H}_m{}^{i+j-k-l}. \tag{17}$$

Equation (17) represents the elements of an influence function matrix, specifically the influence of mode $x^i y^j$ applied to mirror m on the mode $x^k y^l$ measured by beacon b. The tomographic reconstructor can be found by inverting the matrix of the coefficients $\gamma_{kl,b}$ computed for all $\beta_{ij,m}$. Equation (17) holds for both LGSs and NGSs, with the stipulations that for the former, $\gamma_{01,b}$ and $\gamma_{10,b}$ corresponding to global tilt are to be discarded, and for the latter, $h_b \rightarrow \infty$.

B. How Many Beacons Are Required?

Equation (17) describes the field and height dependencies of the beacon measurements on the atmospheric aberration. Aberration of order i produces measured aberrations of order $k \leqslant i$ that vary with the $(i - k)$th power of the field angle. To see how this effect may be used to recover all the needed modal amplitudes, we again construct a simple one-dimensional problem using NGSs for which tilt measurements are valid. In this model, each aberration order i contains only a single mode, namely, x^i, and Eq. (17) becomes

$$\gamma_{k,b} = \sum_{i=k}^{N} \binom{i}{k} \xi_b{}^{i-k} \sum_{m=1}^{M} \beta_{i,m} \hat{H}_m{}^{i-k}, \tag{18}$$

representing a set of polynomials of degree $N - k$ in ξ_b that can be written out as

$$\begin{aligned} \gamma_{1,b} &= a_{10} + a_{11}\xi_b + a_{12}\xi_b{}^2 + \cdots + a_{1(N-1)}\xi_b{}^{N-1}, \\ \gamma_{2,b} &= a_{20} + a_{21}\xi_b + \cdots + a_{2(N-2)}\xi_b{}^{N-2}, \\ &\vdots \\ \gamma_{N,b} &= a_{N0}, \end{aligned} \tag{19}$$

where

$$a_{pq} = \binom{p+q}{q} \sum_{m=1}^{M} \beta_{p+q,m} \hat{H}_m{}^q. \tag{20}$$

Fits to the beacon measurements $\gamma_{k,b}$ will yield the values of a_{pq}. A row of N beacons at distinct ξ_b will then provide sufficient measurements to solve for all distinct $\beta_{i,m}$, that is, for $1 \leqslant m \leqslant i$. Note that the greatest number of constraints on the mirror commands are derived from the lowest-order measurements $\gamma_{1,b}$.

The more general two-dimensional case is similar. For complete characterization of first through Nth atmospheric orders, the most information must again be derived from the measured first-order modes, for which there are also fewest per beacon. Since each order n in two dimensions comprises $n + 1$ modes, the total number of constraints to be derived from these measurements is

$$\sum_{n=1}^{N} n + 1 = \frac{1}{2} N(N + 3), \tag{21}$$

and, since we make two first-order measurements per beacon, we require a minimum of $\frac{1}{4}N(N + 3)$ beacons.

If LGSs are used, we cannot rely on first-order measurements. In the one-dimensional analog above, the first of Eqs. (19) is missing. The total number of atmospheric modal amplitudes that can be recovered from the second-order beacon measurements is then $\frac{1}{2}N(N + 3) - 2 = \frac{1}{2}(N - 1)(N + 4)$. Each beacon provides three second-order measurements, so we need $\frac{1}{6}(N - 1) \times (N + 4)$ beacons.

Without the tilt measurements though, we do not have enough information to solve the tomographic problem. Equation (17) shows that measured modes of a given order are not affected by atmospheric modes of lower order, so it is not helpful to try to make good the missing measurements simply by sensing beacon modes from orders $>N$. Neither is the addition of more LGSs at the same height. As has been previously noted,[11] and is again apparent in Eq. (17), this provides only redundant information. The only recourse then is to use additional beacons at a different height, which may include NGSs.

Many combinations of LGSs and NGSs can be imagined that would provide the needed additional measurements. At one extreme, we could use only NGSs. A sufficiently bright star sensed through order N will yield a solution, at the risk of poor volume filling and resulting aliasing errors. On the other hand, a constellation of $\frac{1}{4}N(N + 3)$ NGSs would also suffice, and only tip and tilt need be measured from each star. This approach is being adopted for the Gemini South MCAO system, which is expected to use measurements from three NGSs within the MCAO field of view.[15] The drawback here is that the need to find three sufficiently bright stars somewhat limits the projected sky coverage of the system.

At the other extreme, one could use just a single NGS for global tilt and a second constellation of LGSs at a different altitude. Hybrid arrangements that use both sodium and Rayleigh LGS (RLGS) have been proposed,[5,16] as well as Rayleigh beacons in which the scattered return from each pulse of laser light is recorded in two or more separate range gates.[7,23] In the latter case, in which the two constellations of LGSs are the same, it is necessary to record all modes in orders 2 through N from one constellation, but it is sufficient to sense only second-order modes from the other.

C. Beacon Geometry

The beacons must be distributed to provide adequate sampling of the atmospheric volume. We therefore seek an orthogonality constraint that guarantees that all modes through order N will generate unique signatures on the WFS. The beacon geometry must allow for the solution of the two-dimensional analog to Eqs. (19). We therefore require that no term in any one of those equations be expressible as a linear combination of other terms; for any nontrivial choice of coefficients c_{ij}, we insist that

$$\sum_{i=0}^{N-1} \sum_{j=0}^{N-i-1} c_{ij} \xi_b{}^i v_b{}^j \neq 0 \quad (22)$$

for at least one beacon b. An arrangement that violates this condition in first order can be constructed by placing all the beacons in a line, obeying $c_{01} v_b + c_{10} \xi_b + c_{00} = 0$ simultaneously. Clearly, such a constellation is unsuited to measurement of modes with curvature in the orthogonal direction. For the case of a single layer of LGSs, the condition is slightly modified because of the lack of first-order wave-front measurements,

$$\sum_{i=0}^{N-2} \sum_{j=0}^{N-i-2} c_{ij} \xi_b{}^i v_b{}^j \neq 0. \quad (23)$$

A beacon geometry that has received some attention recently[5,17] puts four sodium LGSs at the corners of a square, with a fifth at the center. These are supplemented with tilt measurements from three NGSs. The results of Subsection 3.B show that, since the LGSs are all at the same height and the NGSs are not sufficient to recover the quadratic dependence of tilt on field angle, this geometry does not allow a complete determination of third-order modes (just eight of 12 parameters are measurable). Anisoplanatism from third order is then unavoidable. In addition, expression (23) shows that the LGS geometry does not completely sample fourth-order and higher modes, since $\xi_b{}^2 - v_b{}^2 = 0$ for all b. To take full advantage of the compensation available from three DMs, one would need to supplement the beacon measurements. This could be done in a number of ways; the addition of two more NGSs [assuming a constellation of sufficient asymmetry, as determined by expression (22)] or three RLGSs sensing just second order would suffice. Measurement of second-order modes from the original three NGSs would also yield a solution.

4. NUMERICAL SIMULATIONS

The results of the previous sections have not yet been put in the context of a realistic atmospheric turbulence profile. A practical MCAO system with small numbers of DMs and reference beacons will not have enough degrees of freedom either to sense or to correct high-order modes from a real atmosphere perfectly. In this section we model MCAO system performance in response to a realistic profile and illustrate the behavior of the low-order metapupil modes.

A. Calculation of Residual Wave-Front Error

For the purposes of this computation, we model the atmosphere as a set of discrete thin layers of turbulence at distinct heights. We revert to the Zernike basis set to characterize the wave fronts introduced by both the atmospheric layers and the DMs. We then choose DM conjugate heights and a constellation of beacons satisfying the criteria developed in Section 3 and calculate contributions to the residual compensated wave-front error as a function of field angle and mode for a source at range h.

Relying on the transformation of expression 2 and with the aid of Eq. (17), we start by computing a matrix $\mathbf{A}_W$ representing the interaction of each mode of each atmospheric layer on the beacons and corresponding WFSs, which are assumed to be ideal Zernike mode sensors. In an exactly analogous way, we then compute $\mathbf{M}_W$, the interaction matrix coupling the DM modes to the WFSs. A least-squares reconstructor matrix $\mathbf{R}$ is then computed through singular-value decomposition of $\mathbf{M}_W$, allowing for filtering of poorly sensed modes.

Two further interaction matrices are computed, $\mathbf{A}_S(h)$ and $\mathbf{M}_S(h)$ encoding the effect of the atmosphere and DMs, respectively, on the wave front of a source at some chosen position in the field. Together, these matrices encode all the field-dependent effects of Eq. (17). We can then readily calculate a matrix $\mathbf{E}$ of the residual aberrations to be expected in the source wave front for each atmospheric aberration mode:

$$\mathbf{E}(h) = \mathbf{A}_S(h) - \mathbf{M}_S(h)\mathbf{R}\mathbf{A}_W, \quad (24)$$

in which the second term represents the combined effect of the compensation applied by the DMs. The mean-square residual phase error contributed by each mode in the stellar wave front is then given by[24]

$$\boldsymbol{\epsilon}(h) = \mathrm{diag}(\mathbf{EVE}^T), \quad (25)$$

where $\mathbf{V}$ is the atmospheric covariance matrix.[25] The total error summed over all modes is just given by $\sigma^2 = \mathrm{Tr}(\mathbf{EVE}^T)$, a function of the field angle and the MCAO system geometry encoded in $\mathbf{R}$.

B. Description of the Model and Results

Details of the model atmosphere are shown in Table 1, which give a turbulence-weighted mean height $\bar{h}$ of 5090 m and r_0 of 0.15 m at a wavelength of 0.5 μm. This profile reflects preliminary SCIDAR measurements made above Mt. Graham and Mt. Hopkins in Arizona.[26] Atmospheric turbulence was simulated by using 495 Zernike

Table 1. Details of the Model Atmosphere

Height (m)	r_0 (m) at 0.5 μm	Fraction of Total Power
0	0.29	0.34
1800	0.78	0.07
3200	0.41	0.19
5800	0.63	0.09
7400	0.78	0.06
13,100	0.39	0.21
15,800	1.05	0.04

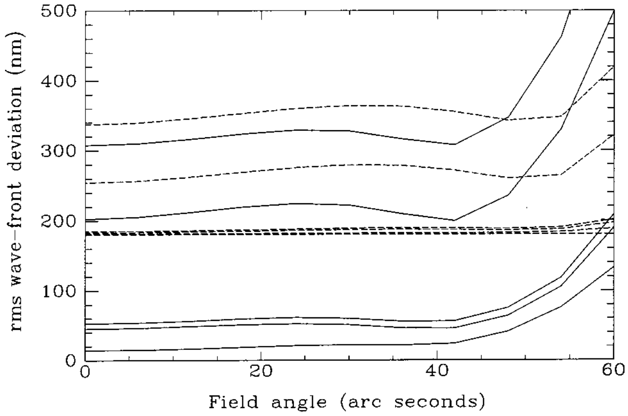

Fig. 2. Contributions to the residual rms wave-front error from modes over the metapupil for stellar sources (solid curves) and sources at 95-km range (dashed curves). For the star, orders 1 and 2 are perfectly corrected, and the plots are coincident with the x axis. The remaining curves, from bottom to top, plot the cumulative contribution from orders 3–5, 13, and 30. For the 95-km objects, tilt introduces 180-nm error, second order is perfectly corrected, and third order is very well corrected below 50 arc sec. The curves plot the contributions from orders 1 and 2, 3, 4, 5, 13, and 30. For both sources, higher orders rapidly degrade the wave front at field angles above ~50 arc sec.

modes per layer, corresponding to radial order 30. Using this model, we have calculated ϵ as a function of field angle for a 10-m telescope with three DMs conjugated to 0, 3.2, and 10 km. Each DM corrected Zernike modes through order 13 over a compensated field of view of 60-arc sec radius. The ground-layer DM was allowed to correct all modes, and the two upper DMs were restricted to orders 2 and above.

The beacon geometry placed five sodium LGSs at 95 km and five RLGSs at 25 km on the vertices of two regular pentagons with radii 60 and 48 arc sec, respectively. The two constellations were clocked by 36° with respect to each other. All Zernike modes in orders 2–13 were sensed from the sodium beacons, whereas only second-order modes were sensed from the RLGSs. Finally, three NGSs measuring just tilt modes were placed in an equilateral triangle on a radius of 30 arc sec. According to the criteria developed in Section 3, this arrangement is sufficient to sense and correct first order for the NGSs, second order for all ranges, and third order at one range. Although the code allows for them, read noise and photon noise were not included in the simulation in order to highlight the effect of anisoplanatic errors.

Results from the simulation are plotted in Fig. 2 for sources at infinity and 95 km, coincident with the LGSs from which most wave-front information was derived. The graphs show the cumulative contributions of the first few aberration orders over the metapupil to the rms residual wave-front deviation σ. In the case of the stars, first-order modes are perfectly corrected, as we expected, since those modes are sensed from NGSs. On the other hand, tip–tilt is not well compensated for the source at 95 km. Indeed, the magnitude of the error is ~180 nm rms, in line with the prediction of Section 2.

In contrast, second-order errors are perfectly corrected at both ranges. Since all three DMs are permitted to compensate second order, these aberrations are corrected throughout the volume. The three independent measurements required to sense these modes are derived from direct second-order measurements from the two sets of LGSs and from the field-dependent tilt of the NGSs. Third-order aberrations, though, are not perfectly compensated for in either case. Here the tomographic information is derived from direct third-order measurements from the sodium LGSs and the linear field dependence of second-order modes from both LGS constellations. Though the geometry is insufficient either to sense or to correct these modes for all ranges, we find that third order is indeed compensated perfectly at a range of approximately 88 km, between the two beacon layers and closer to the sodium layer whence most of the information is derived.

Overall, for sources at both ranges, the residual wave-front error up to order 30 is approximately constant over a field of ~50 arc sec. We find also that, with the exception of tip–tilt noted above, all orders are better compensated in wave fronts from objects at 95 km than those from stars. Total rms residual from orders 2 through 30 is approximately 320 nm for starlight but only 170 nm for the LGS, again because this is the range from which most wave-front information comes. This effect represents generalized focal anisoplanatism that cannot be removed with any number of beacons all at the same finite range.

5. CONCLUSIONS AND DISCUSSION

We have shown that with appropriate wave-front sensing, anisoplanatism in the lowest order, highest-power modes of the atmospheric aberration can be removed across the whole field of view of a MCAO system. This is true regardless of the shape and size of the aperture and, indeed, of the prevailing $C_n{}^2$ profile. As has been noticed by Tokovinin *et al.*,[9] the ability to correct these modes is therefore not compromised by changes to the stratification of turbulence or indeed to changes in elevation angle, which alters the range to vertically localized aberration.

In fact, a number of observations made in that paper on the basis of numerical simulations of a system with two DMs are explained. For example, the mean variance of the residual wave-front error across the field was found to be only a weak function of the conjugation altitude of the modeled system's second DM. This is expected from our result that at any conjugate altitude for that mirror (even negative altitudes), first- and second-order aberrations over the metapupil, containing ~95% of the power in the aberration, will be exactly compensated. The variance of the signals applied to the two DMs was found to be a strong function of the separation between the DMs. The balancing of large signals of opposite sign on the DMs was correctly identified as the source of this variance, but the generalization to an arbitrary number of DMs and a corresponding number of aberration orders was not appreciated.

We note in passing that the ability to correct low-order metapupil modes with no isoplanatic error suggests a way to improve on standard methods of computing tomographic reconstructors. Current methods typically invert an influence function such as Eq. (17), taking into account known statistical properties of the atmosphere and noise in the WFS measurements. In the future we plan to investigate the removal of anisoplanatism as an additional constraint in the inversion.

Compensation of low-order aberration with multiple DMs is likely to lead to an excess of controllable modes, giving rise to degeneracies in the reconstructor matrix. These modes, which are completely identifiable, must be explicitly removed in the computation of the matrix to avoid instability in the control loop.

Necessary and sufficient conditions on the number and placement of beacons have been found that must be satisfied if atmospheric aberration up to a given order is to be completely sensed. These conditions, in variants appropriate for NGSs and LGSs, lead to a natural relationship between the number of DMs in a MCAO system and the minimum number of beacons. We have illustrated these analytic results with simulations that implement a deterministic calculation of the residual wave-front error as a function of field angle.

In wave fronts recovered from LGSs, we have identified a contribution to focal anisoplanatism that is unrelated to volume filling. Height diversity in the reference beacons is the only way that errors in the tomographic reconstruction from this anisoplanatism can be avoided. Furthermore, corrections not subject to this error can be applied only for aberration orders less than the number of DMs. Height diversity may be provided by multiple NGSs and a single layer of LGSs, or constellations of LGSs at two altitudes, which can be achieved in a number of ways. Multiple exposures of the same pulse of beacon light from different heights, either with dynamic refocus[16] or as input to a phase-diverse algorithm,[23] are two possibilities. A third is the use of both Rayleigh LGSs and sodium LGSs.

Several important consequences arise from this focal anisoplanatism. In the first place, the DM commands to minimize residual aberration in sources at infinity are not those that minimize aberration in the LGSs and therefore drive the WFS signals to null. In addition, if viable schemes are ever implemented to sense global image motion from a LGS, it will no longer be sufficient to use a single tip–tilt mirror in the compensating optics, since tilt errors sensed by starlight and laser light will differ by a substantial amount. Without an explicit knowledge of the height dependence of the tilt error, it will be essential to drive two mirrors in tilt to correct objects at all ranges simultaneously.

ACKNOWLEDGMENTS

This study has been supported by grants from the U.S. Air Force Office of Scientific Research and the National Science Foundation.

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REFERENCES

1. J. R. P. Angel, M. Lloyd-Hart, E. K. Hege, R. J. Sarlot, and C. Peng, "The 20/20 telescope: MCAO imaging at the individual and combined foci," in *Beyond Conventional Adaptive Optics*, R. Ragazzoni, ed. (Astronomical Observatory and Department of Astronomy, Padova, Italy, 2001). Available on line at http://www.adaopt.it/venice2001/proceedings/pdf/angel_pap.pdf, 2001.
2. J. E. Nelson, "Design concepts for the California Extremely Large Telescope (CELT)," in *Telescope Structures, Enclosures, Controls, Assembly/Integration/Validation, and Commissioning*, T. A. Sebring and T. Andersen, eds., Proc. SPIE **4004**, 282–289 (2000).
3. P. Dierickx and R. Gilmozzi, "Progress of the OWL 100-m telescope conceptual design," in *Telescope Structures, Enclosures, Controls, Assembly/Integration/Validation, and Commissioning*, T. A. Sebring and T. Andersen, eds., Proc. SPIE **4004**, 290–299 (2000).
4. J. M. Beckers, "Increasing the size of the isoplanatic patch with multiconjugate adaptive optics," in *ESO Conference on Very Large Telescopes and Their Instrumentation*, M.-H. Ulrich, ed. (European Southern Observatory, Garching bei München, Germany, 1998), Vol. 2, pp. 693–703.
5. I. De La Rue and B. L. Ellerbroek, "Multiconjugate adaptive optics with hybrid laser beacon systems," in *Adaptive Optics Systems and Technology II*, R. K. Tyson, D. Bonaccini, and M. C. Roggemann, eds., Proc. SPIE **4494**, 290–301 (2002).
6. D. Bonaccini, W. Hackenberg, R. Davies, M. Cullum, S. Rabien, E. Brunetto, T. Ott, M. Quattri, S. Hippler, E. Allaert, M. Dimmler, M. Tarenghi, A. van Kesteren, C. di Chirico, B. Buzzoni, P. Gray, and U. Neumann, "VLT Laser Guide Star Facility: from one to many LGS," in *Beyond Conventional Adaptive Optics*, R. Ragazzoni, ed. (Astronomical Observatory and Department of Astronomy, Padova, Italy, 2001). Available on line at http://www.adaopt.it/venice2001/proceedings/pdf/bonaccini_pap.pdf, 2001.
7. M. Lloyd-Hart and N. M. Milton, "Design and expected performance of the 6.5 m MMT MCAO system," in *Adaptive Optical System Technologies II*, P. L. Wizinowich and D. Bonaccini, eds., Proc. SPIE **4839**, 578–587 (2003).
8. A. Tokovinin, M. Le Louarn, and M. Sarazin, "Isoplanatism in a multiconjugate adaptive optics system," J. Opt. Soc. Am. A **17**, 1819–1827 (2000).
9. A. Tokovinin, M. Le Louarn, E. Viard, N. Hubin, and R. Conan, "Optimized modal tomography in adaptive optics," Astron. Astrophys. **378**, 710–721 (2001).
10. M. Le Louarn and M. Tallon, "Analysis of modes and behavior of a multiconjugate adaptive optics system," J. Opt. Soc. Am. A **19**, 912–925 (2002).
11. B. L. Ellerbroek and F. J. Rigaut, "Methods for correcting tilt anisoplanatism in laser-guide-star-based multiconjugate adaptive optics," J. Opt. Soc. Am. A **18**, 2539–2547 (2001).
12. R. C. Flicker, "Sequence of phase correction in multiconjugate adaptive optics," Opt. Lett. **26**, 1743–1745 (2001).
13. M. C. Roggemann and A. C. Koivunen, "Branch-point reconstruction in laser beam projection through turbulence with finite-degree-of-freedom phase-only wave-front correction," J. Opt. Soc. Am. A **17**, 53–62 (2000).
14. R. Ragazzoni, E. Marchetti, and F. Rigaut, "Modal tomography for adaptive optics," Astron. Astrophys. **342**, L53–L56 (1999).
15. F. J. Rigaut, B. L. Ellerbroek, and R. Flicker, "Principles, limitations, and performance of multiconjugate adaptive optics," in *Adaptive Optical Systems Technology*, P. L. Wizinowich, ed., Proc. SPIE **4007**, 1022–1031 (2000).
16. R. Angel and M. Lloyd-Hart, "Atmospheric tomography with Rayleigh laser beacons for correction of wide fields and 30-m-class telescopes," in *Adaptive Optical Systems Technology*, P. L. Wizinowich, ed., Proc. SPIE **4007**, 270–276 (2000).
17. R. Flicker, F. J. Rigaut, and B. L. Ellerbroek, "Comparison of multiconjugate adaptive optics configurations and control algorithms for the Gemini South 8-m telescope," in *Adaptive Optical Systems Technology*, P. L. Wizinowich, ed., Proc. SPIE **4007**, 1032–1043 (2000).
18. E. Viard, N. N. Hubin, and M. Le Louarn, "Adaptive optics with four laser guide stars: cone effect correction on large telescopes," in *Adaptive Optical Systems*

Technology, P. L. Wizinowich, ed., Proc. SPIE **4007**, 106–114 (2000).

19. R. Ragazzoni, “Robust tilt determination from laser guide stars using a combination of different techniques,” Astron. Astrophys. **319**, L9–L12 (1997).
20. M. Schöck, R. Foy, M. Tallon, L. Noethe, and J.-P. Pique, “Performance analysis of polychromatic laser guide stars used for wavefront tilt sensing,” Mon. Not. R. Astron. Soc. **337**, 910–920 (2002).
21. M. S. Belen'kii, S. J. Karis, J. M. Brown II, and R. Q. Fugate, “Experimental validation of a technique to measure tilt from a laser guide star,” Opt. Lett. **24**, 637–639 (1999).
22. S. Esposito, R. Ragazzoni, A. Riccardi, C. O'Sullivan, N. Ageorges, M. Redfern, and R. Davies, “Absolute tilt from a laser guide star: a first experiment,” Exp. Astron. **10**, 135–145 (2000).
23. M. Lloyd-Hart, S. M. Jefferies, J. R. P. Angel, and E. K. Hege, “Wave-front sensing with time-of-flight phase diversity,” Opt. Lett. **26**, 402–404 (2001).
24. G. Rousset, “Wave-front sensors,” in *Adaptive Optics in Astronomy*, ed. F. Roddier (Oxford U. Press, New York, 1998), p. 118.
25. R. J. Noll, “Zernike polynomials and atmospheric turbulence,” J. Opt. Soc. Am. **66**, 207–211 (1976).
26. D. L. McKenna, R. Avila, J. M. Hill, S. Hippler, P. Salinari, P. C. Stanton, and R. Weiss, “LBT facility SCIDAR: recent results,” in *Adaptive Optical System Technologies II*, P. L. Wizinowich and D. Bonaccini, eds., Proc. SPIE **4839**, 825–836 (2003).